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THE SECOND PLAYER'S STRATEGY IN A LINEAR DIFFERENTIAL GAME*

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A linear antagonistic two-person differential game with fixed instant of termination and convex pay-off function is considered. A numerical method is described for constructing the second (maximizing) player's strategy, which guarantees a close-to-optimal result. A computer-checked example is given. The paper is related to /1-8/.

Among antagonistic two-person differential games with geometric constraints on the control parameters /1/, the simplest from the point of view of numerical solution are games with linear dynamics, a fixed instant of termination, and convex terminal pay-off function /3-8/. In many problems the pay-off function depends on some, say n, but not all, the coordinates of the phase vector. This feature can be used to reduce the dimensionality of the problem by passing to an equivalent n-th order game /1/.

For the case $n = 2, 3$, algorithms have now been developed and computerized for constructing the sets of levels $W^\circ(\cdot, c) = \{(t, y): \Gamma^\circ(t, y) \leqslant c\}$ of the pay-off function Γ° of the equivalent game /9/. Set $W^\circ(\cdot, c)$ is otherwise defined as the first (minimizing) player's maximum stable bridge in the equivalent game, which breaks off at the instant of termination ϑ in the level set $\gamma \leqslant c$ of pay-off function γ /1/. Its section $W^\circ(t, c)$ at instant t is the same as the alternated integral /2/, constructed in the interval $[t, \vartheta]$ of the set $\gamma \leqslant c$. Introducing the mesh $\{c_j\}$ and constructing the set $\{W^\circ(\cdot, c_j)\}$, we can find by computer the optimal (or near-optimal) strategies in the initial problem.

Since, given any c and t, sections $W^\circ(t, c)$ are convex, as the basis for constructing the first player's stable optimal strategy we can take the strategy of extremal aiming /1/. The calculations are simplified if the first player's control parameter is scalar: the optimal strategy is in this case given with the aid of a switching surface /8/. The non-convexity of the complement $R^n \setminus W^\circ(t, c)$ prevents similar methods being used to find the second player's stable optimal strategy. The attempt to use another well-known metod of general type, namely, control with leading guide, also encounters difficulties, in that the problem of forming a stable guidance motion arises.

Our present aim is to give a numerical method for constructing a stable quasi-optimal second player's strategy for linear differential games with fixed instant of termination. The method is based on a preliminary determination of sections $W^\circ(t, c)$ and is primarily aimed at problems with $n = 2, 3$ and a set Q, bounding the second player's control parameter, consisting of a polyhedron with a small number of vertices. For such problems, the method is realized as a standard computer program.

1. Formulation of the problem. We take the linear antagonistic two-person differential game

$$\begin{gathered} z^{\cdot} = A(t) z + B(t) u + C(t) v \\ z \in R^m, u \in P, v \in Q, t \in T = [t^0, \vartheta] \end{gathered} \tag{1.1}$$

with fixed instant ϑ of termination. The matrix $A(t)$ is continuous, and the matrices $B(t)$, $C(t)$ satisfy a Lipschitz condition with respect to t. The sets P and Q are respectively a convex compactum and convex polyhedron in finite-dimensional spaces. The polyhedron Q may be degenerate. We assume that the convex pay-off function γ depends only on some $n\,(1 < n \leqslant m)$ coordinates of the phase vector. Let these be the first n coordinates. We additionally assume that, given any c, the set of levels $M(c) = \{(z_1, \ldots, z_n)' : \gamma(z_1, \ldots, z_n) \leqslant c\}$ of the function γ is bounded. The first player has a parameter u at his disposal and minimizes $\gamma(z_1(\vartheta), \ldots, z_n(\vartheta))$. The second player has a parameter v at his disposal and has the opposite aims.

Let E be a compactum in $T \times R^n$. We wish to construct a stable second player's strategy which guarantees him, for all initial positions (t_*, z_*) of E, a result close to the value of the game.

2. Polyhedron $W(t_i, c)$. Let $Z_n(\vartheta, t)$ be the matrix of the first n rows of the fundamental Cauchy matrix for system $z^{\cdot} = A(t) z$. Using the replacement $y(t) = Z_n(\vartheta, t) z(t)$, we transform

*Prikl.Matem.Mekhan., 51, 2, 193-200, 1987 from (1.1) to the equivalent game

$$y^{\cdot} = B^1(t)u + C^1(t)v \tag{2.1}$$
$$B^1(t) = Z_n(\vartheta, t)B(t), \ C^1(t) = Z_n(\vartheta, t)C(t)$$
$$y \in R^n, \ u \in P, \ v \in Q, \ t \in T$$

of n-th order with the previous pay-off function γ. We divide the interval T with a step $\varkappa$ by points $t_0 = t^0, t_1, t_2, \ldots$. We put $B^2(t) = B^1(t_i), C^2(t) = C^1(t_i), \ t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots$

Let P^2 be the convex polyhedron approximating compactum P, and γ^2 the convex function with polyhedral level sets $M^2(c)$, close to function γ. We replace system (2.1) by the game

$$y^{\cdot} = B^2(t)u + C^2(t)v, \ y \in R^n, \ u \in P^2, \ v \in Q, \ t \in T \tag{2.2}$$

with piecewise constant dynamics. For all t_i, the sections $W(t_i, c)$ of the level set of the value function in game (2.2) are convex polyhedra. Cases when the polyhedra degenerate are possible. This leads to instability of the numerical procedure for constructing them. We shall assume below that polyhedra $W(t_i, c)$ are non-degenerate. These polyhedra approximate the sections $W^\circ(t_i, c)$ of the level set of the value function in game (2.1) /4, 7/.

Let us describe the construction of polyhedron $W(t_i, c)$. Let ρ be the symbol of the support function, $N(X)$ the set of all unit outward normals to $n-1$ -dimensional faces of the polyhedron $X \subset R^n$. We put $W(\vartheta, c) = M^2(c)$. Assume that polyhedra $W(\vartheta, c), \ldots, W(t_{i+2}, c), W(t_{i+1}, c)$ have been found. We take $L(t_i, c) = N(W(t_{i+1}, c) - \varkappa B^2(t_i)P^2)$ and

$$\eta(l, t_i, c) = \rho(l, W(t_{i+1}, c)) + \varkappa\rho(l, -B^2(t_i)P^2) - \varkappa\rho(l, C^2(t_i)Q), \ l \in L(t_i, c) \tag{2.3}$$

Then, /10/,

$$W(t_i, c) = \{y \in R^n : l'y \leqslant \eta(l, t_i, c), \ l \in L(t_i, c)\} \tag{2.4}$$

We have $L(t_i, c) \supset N(W(t_i, c))$. In many problems the set $L(t_i, c)$ increases rapidly as i decreases. Hence, for computer evaluations, we need to limit the number of vectors included in $L(t_i, c)$, by "pasting together" close vectors. Accordingly, instead of $W(t_i, c)$ we obtain an upper bound $\mathbf{W}(t_i, c)$. To cover this case, we introduce for $t_i \neq \vartheta$ an arbitrary set $\mathbf{L}(t_i, c)$ of unit vectors, which satisfy the condition $\mathbf{L}(t_i, c) \supset N(\mathbf{W}(t_{i+1}, c))$. We put $\mathbf{W}(\vartheta, c) = M^2(c)$. For $t_i \neq \vartheta$, we find $\eta(l, t_i, c)$ and polyhedron $\mathbf{W}(t_i, c)$ from expressions similar to (2.3), (2.4), with $L(t_i, c)$ replaced by $\mathbf{L}(t_i, c)$ and $W(t_{i+1}, c)$ by $\mathbf{W}(t_{i+1}, c)$. Since $N(\mathbf{W}(t_i, c)) \subset \mathbf{L}(t_i, c)$, the set $\mathbf{L}(t_i, c)$ in particular can be taken as constant, independent of t_i. We require only that it contain all the normals of polyhedron $M^2(c)$.

We fix the interval $[c^\circ, c^*]$. We stipulate that c° is a lower bound of the pay-off function in the set of initial positions in game (1.1), and c^* is not less than the analogous upper bound. In the interval $[c^\circ, c^*]$ we assign a mesh $\{c_j\}$ of values $c_1 < c_2 < \ldots$ of the parameter c. For each $c \in \{c_j\}$, let a sequence of polyhedra $\mathbf{W}(t_i, c)$ be computer-constructed. Let r° and r^* denote positive numbers such that, in any of polyhedra $\mathbf{W}(t_i, c)$ we can inscribe a sphere of radius r° and circumscribe any by a sphere of radius r^*. The centres of these spheres may depend on c and t_i.

The support function of polyhedron $\mathbf{W}(t_i, c)$ is piecewise linear. In fact, to any view w there corresponds a convex cone, generated by the unit outward normals to $n-1$ -dimensional faces, containing w. In the cone with function $\rho(\cdot, \mathbf{W}(t_i, c))$ is linear, while the unit normals to the faces are the cone generators. When $n = 2$ there are two generators, and with $n \geqslant 3$ there are not less than n. We stipulate that, with $n \geqslant 3$, every cone of linearity of the function $\rho(\cdot, \mathbf{W}(t_i, c))$ may be divided without the addition of new generators into n-faced cones. When using the expression "cone of linearity of the function $\rho(\cdot, \mathbf{W}(t_i, c))$" we understand that the number of its generators is n. We assume the existence of a constant $\beta > 0$, common for all $c \in \{c_j\}$ and t_i, such that, for any cone of linearity $K = \operatorname{cone}\{l_1, \ldots, l_n\}$, of the function $\rho(\cdot, \mathbf{W}(t_i, c))$ we have

$$l_s - l_k \mid \mid D_{se} \mid / \mid D \mid \leqslant \beta, \ s, k, e = 1, 2, \ldots, n \tag{2.5}$$

Here, D is the determinant of the matrix composed of the coordinates of vectors $l_1, \ldots, l_n$ of the cone generators, D_{se} is the cofactor of the (s, e)-th element. When $n = 2$ the condition is satisfied **automatically**: the constant β can be taken equal to $2r^*/r^\circ$. With $n = 3$, the condition limits the "elongation" of the cones of linearity of the function $\rho(\cdot, \mathbf{W}(t_i, c))^1$.* (*M.A. Zarkh and V.S. Patsko, The second player's positional control in a linear differential game with fixed instant of termination, Dep. VINITI, 5756,-85, 1 August, 1985).

3. **Processing of the polyhedra $\mathbf{W}(t_i, c)$.** Let $Q(l, t_i)$ be the set of all vertices of polyhedron Q, on which the maximum of the scalar product $l'C^2(t_i)q, q \in Q$, is reached. We denote by $\Lambda(t_i, c)$ the union of all cones of linearity $K = \operatorname{cone}\{l_1, \ldots, l_n\}$ of function

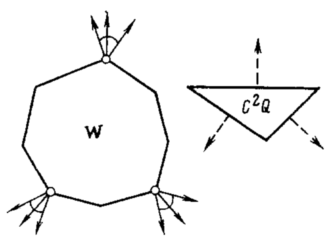

Fig.1

$\rho(\cdot, \mathbf{W}(t_i, c))$, for each of which $\bigcap_s Q(l_s, t_i) = \varnothing, s = \overline{1, n}$. The cones appearing in $\Lambda(t_i, c)$ will be called "poor". In Fig.1, for $n = 2$, we show the polyhedra $\mathbf{W}(t_i, c)$, $C^2(t_i)Q$ (denoted $\mathbf{W}$ and C^2Q). The poor cones of linearity of $\rho(\cdot, \mathbf{W}(t_i, c))$ are those into whose interior the normals of polyhedron $C^2(t_i)Q$ are incident.

Of the information about polyhedra $\mathbf{W}(t_i, c)$ we retain, for all $c \in \{c_j\}$ and t_i for the formation of the second player's strategy, only the information about the poor cones and the values of support function $\rho(\cdot, \mathbf{W}(t_i, c))$ on their generators. The control algorithm that uses this information will be called an algorithm with correction.

We stipulate that the initial instant t_* for game (1.1) is the same as one of the instants t_i. We assume that the step Δ of the second player's control scheme /1/ is constant and a multiple of $\varkappa$. Let $\tau_0 = t_*, \tau_k = \tau_{k-1} + \Delta, k = 1, 2, \ldots$. When describing the algorithm with correction we shall assume that, instead of the exact phase vector $z(\tau_k)$, the reading $\xi(\tau_k)$ is transmitted in the second player's control scheme. Before the algorithm starts to operate, we must specify the parameter $\mu, 0 < \mu < 1$, the meaning of which will be clear from what follows.

Given any unit vector $l \in R^n$, any $c \in \{c_j\}, y \in R^n$, and t_i, we put $d(t_i, l, y, c) = \rho(l, \mathbf{W}(t_i, c)) - l'y$. The quantity d is the distance of point y to the corresponding vector l of the support hyperplane to polyhedron $\mathbf{W}(t_i, c)$. Note that d has a plus sign if y belongs to the same half-space as $\mathbf{W}(t_i, c)$, and a minus sign if y and $\mathbf{W}(t_i, c)$ are separated by this hyperplane.

4. Basic algorithm. The algorithm with correction is based on the so-called basic algorithm, which uses information taken from polyhedra $\mathbf{W}(t_i, c)$ with fixed c, and will now be described.

We fix $c \in \{c_j\}$. Assume that $\mu, 0 < \mu < 1$, has been chosen. At each instant τ_k the second player's control is found from the extremum condition on some vector. At the instant $t_* = \tau_0$ we have to specify the initial unit vector $l_* = l(\tau_0)$.

In short, at the instant t_* we have the reading $\xi(t_*)$ and the vector l_*. To the instant $\tau_k, k \geqslant 0$, there corresponds the reading $\xi(\tau_k)$ and the unit vector $l(\tau_k)$, transmitted from the previous step of the discrete scheme. If $l(\tau_k) \notin \Lambda(\tau_k, c)$, we put $l(\tau_{k+1}) = l(\tau_k)$. Let $l(\tau_k) \in \Lambda(\tau_k, c)$, i.e., a poor cone $K = \operatorname{cone}\{l_1, \ldots, l_n\}$, containing the vector $l(\tau_k)$, is discovered. If the inequalities $l'(\tau_k) l_s \leqslant 1 - \mu, s = 1, 2, \ldots, n$ hold simultaneously, we choose as $l(\tau_{k+1})$ the generator $l_1, \ldots, l_n$ of the cone K, in which the minimum of distances $d(\tau_k, l_s, Z_n(\vartheta, \tau_k)\xi(\tau_k), c), s = 1, 2, \ldots, n$, is reached. If $l'(\tau_k) l_s > 1 - \mu$, for at any rate one $s = 1, 2 \ldots, n$, we take $l(\tau_{k+1}) = l(\tau_k)$. As the second player's control signal in the interval $[\tau_k, \tau_{k+1})$ we take any vector of $Q(l(\tau_{k+1}), \tau_k)$.

The essence of the algorithm is briefly as follows. The second player's control, chosen in the interval $[\tau_k, \tau_{k+1})$, tries to repel system (2.1) at instant τ_k from polyhedron $\mathbf{W}(\tau_k, c)$ along the vector $l(\tau_{k+1})$. If the vector $l(\tau_{k+1})$ hits at the instant τ_{k+1} a poor cone, at a reasonable distance (defined by μ) from its generators, the direction of further repulsion is changed. Otherwise it remains as before. The information on the state reading of system (1.1) is used only at instants when the direction of repulsion changes.

Let φ be an estimate of the accuracy of readings $\xi(\tau_k)$: $|z(\tau_k) - \xi(\tau_k)| \leqslant \varphi$ for any k. We put $\| Z_n(\vartheta, \cdot) \| = \max |Z_n(\vartheta, t) x|$. The maximum is taken over all unit vectors x and all $t \in T$. We take $\alpha = \| Z_n(\vartheta, \cdot) \| \varphi$. Let g denote the upper bound of the modulus of the right-hand side of system (2.2) in T.

We assume that $\varkappa g \leqslant r^\circ/2$. In the interval $[t_*, \vartheta]$, let the second player's control be obtained according to the basic algorithm. Then, for any realization of the first player's control, we have the estimate

$$\begin{aligned} d(\vartheta, l(\vartheta), Z_n(\vartheta, \vartheta) z(\vartheta), c) \leqslant \\ \max\{2\alpha/\mu, d(t_*, l_*, Z_n(\vartheta, t_*)\xi(t_*), c)\} + 2\alpha + \\ \sigma_1(\vartheta - t_*)\Delta + \sigma_2(\vartheta - t_*)\sqrt{\mu} + \varepsilon(t_*, \vartheta) \end{aligned} \tag{4.1}$$

Here, σ_1, σ_2 are positive constants, dependent only on the form of system (2.1) and on the constants r°, r^*, β. The quantity $\varepsilon(t_*, \vartheta)$ is given by the relation

$$\begin{aligned} \varepsilon(t_*, \vartheta) = \int_{t_*}^{\vartheta} \max_{|l| \leqslant 1} [\rho(l, -B^1(\tau)P) - \rho(l, -B^2(\tau)P^2) + \\ \rho(l, (C^2(\tau) - C^1(\tau))Q)]\, d\tau \end{aligned}$$

and characterizes the difference in the dynamics of systems (2.1) and (2.2).

Let F denote the right-hand side of (4.1). Let $J \subset R^n$ be a compactum, upper-bounding the possible positions of the vector $(z_1(\vartheta), \ldots, z_n(\vartheta))'$ (when the initial positions are taken in E), $\| \gamma - \gamma^2 \|_J$ is the norm of the difference between the functions γ and γ^2 in J, ζ is the

Lipschitz constant of the function γ^2 in $M^2(c^*)$. Since the Euclidean distance from the point $Z_n(\vartheta, \vartheta)\, z(\vartheta) = (z_1(\vartheta), \ldots, z_n(\vartheta))'$ to the set $\{y \in R^n : \gamma^2(y) \geqslant c\}$ does not exceed $d(\vartheta, l(\vartheta), Z_n(\vartheta, \vartheta)\, z(\vartheta), c)$, we have from (4.1):

$$\gamma(z_1(\vartheta), \ldots, z_n(\vartheta)) \geqslant c - \zeta F - \|\gamma - \gamma^2\|_J \tag{4.2}$$

Inequality (4.2) describes the second player's guarantee with control by the basic algorithm. The right-hand side of the inequality is close to $c - \zeta\sigma_2(\vartheta - t_*)\sqrt{\mu}$, if the games (2.1), (2.2) are reasonably similar, the step Δ is small, and the initial distance $d(t_*, l_*, Z_n(\vartheta, t_*)\,\xi(t_*), c)$ and φ are small. If, moreover, μ is small, and c differs little from the game value in position $(t_*, \xi(t_*))$, then the second player's guarantee is close to optimal.

The strategy defined by the basic algorithm is stable with respect to inaccuracies of the state reading $z(\tau_k)$. We also have stability with respect to small errors in constructing polyhedra $\mathbf{W}(t_i, c)$. This can be proved on the basis of the claims used in the proof of inequality (4.1).

Notes. 1°. Let τ_e be an instant when $l(\tau_e)$ changes into a new and different vector $l(\tau_{e+1})$. Knowing $l(\tau_{e+1})$, we can read at instant τ_e the first instant $\tau_h, h > e$, when the vector $l(\tau_h) = l(\tau_{e+1})$ hits a poor cone and is replaced by a new one. Hence, at the instant τ_e we can form a second player's peicewise constant programmed control in system (1.1) throughout the interval $[\tau_e, \tau_h)$. This programmed control satisfies at instants τ_k, $e \leqslant k < h$, Pontryagin's maximum condition on the function $\psi(t) = Z_n'(\vartheta, t)\, l(\tau_{e+1})$. At the instant τ_h, using reading $\xi(\tau_h)$, we choose a new vector $l(\tau_{h+1})$, find the interval $[\tau_h, \tau_s)$, $s > h$, in which the vector does not change, and form the programmed control in $[\tau_h, \tau_s)$. At instants τ_k, $h \leqslant k < s$, it satisfies the maximum condition on the function $\psi(t) = Z_n'(\vartheta, t)\, l(\tau_{h+1})$, etc. Thus, the second player's control defined by the basic algorithm can be realized as a piecewise programmed control.

2°. It may happen that, for all instants τ_k the set $\Lambda(\tau_k, c)$ formed from poor cones is empty. This is the case e.g., when it is assumed, when constructing polyhedra $\mathbf{W}(t_i, c)$, that $\mathbf{L}(t_i, c) = N(\mathbf{W}(t_{i+1}, c) - \varkappa B^2(t_i) P^2)$ (consequently, $\mathbf{W}(t_i, c) = W(t_i, c)$) and we have the one-type property /1/: $B^2 = C^2$, $P^2 = -\lambda Q$, $\lambda > 1$. In this case $l(\tau_k) = l_*$ for any k, so that the second player's control, defined by the basic algorithm, can be specified at the initial instant t_* throughout the interval $[t_*, \vartheta]$ by a single program, which satisfies at instants τ_k the maximum condition on the function $\psi(t) = Z_n'(\vartheta, t)\, l_*$.

5. Algorithm with correction.

The general scheme of the algorithm is as follows. We fix the parameter μ. At the instant t_*, using the reading $\xi(t_*)$, we in some way choose the initial value $c_* \in \{c_j\}$ and the initial vector l_*. We form the second player's control in accordance with the basic algorithm with $c = c_{j(0)} = c_*$ with step Δ in the interval $[\tau_{k(0)}, \tau_{k(1)})$, where $\tau_{k(0)} = \tau_0 = t_*$, and $\tau_{k(1)} = t_* + k(1)\Delta$ the instant of first correction. The correction at instant $\tau_{k(1)}$ amounts to readjustment of the basic algorithm to the new value $c_{j(1)} \geqslant c_{j(0)}$, $c_{j(1)} \in \{c_j\}$ of the parameter c. The control according to the basic algorithm is now realized with $c = c_{j(1)}$ in $[\tau_{k(1)}, \tau_{k(2)})$, where $\tau_{k(2)} = \tau_{k(1)} + (k(2) - k(1))\Delta$ is the instant of second correction, etc. The choice of c_*, l_*, and of the instants of correction $\{\tau_{k(p)}\}$, and the correction itself at the instant $\tau_{k(p)}$, connected with transition to the value $c_{j(p)} \geqslant c_{j(p-1)}$, $c_{j(p)} \in \{c_j\}$, may be found by different methods. Let us describe one version.

Let $\Omega(t_i, c)$ be the set of all generators of poor cones of $\Lambda(t_i, c)$. We put $\mathbf{H}(t_i, c) = \{y \in R^n : l'x \leqslant \rho(l, \mathbf{W}(t_i, c)), l \in \Omega(t_i, c)\}$. We have $\mathbf{H}(t_i, c) \supset \mathbf{W}(t_i, c)$, $N(\mathbf{H}(t_i, c)) = \Omega(t_i, c)$. Data about the inequalities that define polyhedra $\mathbf{H}(t_i, c)$ are contained in the information prepared for the algorithm with correction. As c_* we take the maximum of the values $c \in \{c_j\}$, for which $Z_n(\vartheta, t_*)\,\xi(t_*) \notin \operatorname{int} \mathbf{H}(t_*, c)$. If there are no values with this property, we put $c_* = c_1$. We define l_* as the vector from the set $N(\mathbf{H}(t_*, c_*))$, which minimizes $d(t_*, l, Z_n(\vartheta, t_*)\,\xi(t_*), c_*)$. We specify the instant of correction $\tau_{k(p)}$, $p \geqslant 1$, as the first instant $\tau_k > \tau_{k(p-1)}$, when the vector $l(\tau_k)$, transmitted from the previous step of the discrete scheme, hits one of the poor cones $K = \operatorname{cone}\{l_1, \ldots, l_n\} \subset \Lambda(\tau_k, c_{j(p-1)})$, where $l'(\tau_k)\, l_s \leqslant 1 - \mu$ for all $s = 1, 2, \ldots, n$. At the instant $\tau_{k(p)}$ we have the reading $\xi(\tau_{k(p)})$, the vector $l(\tau_{k(p)})$, and the value $c_{j(p-1)}$. As $c_{j(p)}$ we take the maximum of the values $c \in \{c_j\}$, $c \geqslant c_{j(p-1)}$, for which $Z_n(\vartheta, \tau_{k(p)})\,\xi(\tau_{k(p)}) \notin \operatorname{int} \mathbf{H}(\tau_{k(p)}, c)$. If there is no value with this property, we put $c_{j(p)} = c_{j(p-1)}$. When realizing the first version of choosing $c_{j(p)}$, we define $l(\tau_{k(p)+1})$ at the next step of the discrete scheme as the vector of the set $N(\mathbf{H}(\tau_{k(p)}, c_{j(p)}))$, which minimizes $d(\tau_{k(p)}, l, Z_n(\vartheta, \tau_{k(p)})\,\xi(\tau_{k(p)}), c_{j(p)})$. In the second case we specify $l(\tau_{k(p)+1})$ as the usual next vector of the basic algorithm with $c = c_{j(p-1)} = c_{j(p)}$. Having chosen this vector, we perform all the operations provided by the basic algorithm with $c = c_{j(p)}$ until the next instant of correction.

Transition to new values of c during the game is introduced into the algorithm with correction in order to allow for "non-optimality" of the first player's behaviour. When his behaviour is optimal, the result largely depends on the choice of c_* and l_*. The second player's guarantee is best if c_* is close to the game value in the position $(t_*, \xi(t_*))$, and l_* is close to the vector that minimizes the distance

$$d(t_*, l, Z_n(\vartheta, t_*)\,\xi(t_*), c_*) = l' Z_n(\vartheta, t_*)\,\xi(t_*) - \rho(l, \mathbf{W}(t_*, c_*))$$

in the set of unit vectors $l \in R^n$. With this method of choosing c_* and l_* on the basis of information about polyhedra $\mathrm{H}(t_*, c), c \in \{c_j\}$, this is not necessarily the case. Hence it is best to store for instants t_* (specially when the initial instant in the problem is always the same), complete information about the polyhedra $\mathrm{W}(t_*, c), c \in \{c_j\}$, and to use it to specify c_*, l_*, by changing $\mathrm{H}(t_*, c)$ into $\mathrm{W}(t_*, c)$. In the case of a sufficently coarse mesh $\{c_j\}$, the second player's guarantee will then be close to optimal and the corresponding strategy may be called quasi-optimal. The strategy is stable.

Results of computing specific examples show that, for many problems, there is no sense in taking a large number of values c in mesh $\{c_j\}$. Satisfactory results are obtained even when only one - three values of c are chosen.

6. Examples. We shall use the above method of constructing the second player's control to form the worst wind disturbances according to the feedback principle in the problem of aircraft control in the vertical plane on landing. The statement of the problem is due to V.M. Kein and A.I. Krasov.

The differential equations of motion of the aircraft centre of mass in the vertical plane in the neighbourhood of the nominal trajectory, linearized on the assumption of constant path velocity and thrust force, are

$$\begin{gathered} z_1^{\cdot} = z_2, \quad z_2^{\cdot} = -0{,}695z_2 + 0{,}91z_3 + 0{,}26z_6 + 0{,}695z_7 \\ z_3^{\cdot} = z_4, \; z_4^{\cdot} = 0{,}616z_2 - 0{,}806z_3 - 0{,}676z_4 - 0{,}419z_5 - 0{,}616z_7 \\ z_5^{\cdot} = -4z_5 + 4u, \quad z_6^{\cdot} = -0{,}5z_6 + 0{,}5v_1, \quad z_7^{\cdot} = -0{,}5z_7 + 0{,}5v_2 \\ |u| \leqslant 20, \quad |v_1| \leqslant 10, \quad |v_2| \leqslant 5, \quad t \in T = [0, 15] \end{gathered} \tag{6.1}$$

The coordinates z_1, z_3 are the altitude and pitching angle deviations, z_2 and z_4 are the rates of deviation, and z_5 is the elevator deviation angle. The height deviation is measured in metres, the angles in degrees, and the time in seconds. The variation of the coordinate z_5 is given by the fifth equation, and parameter u is the control wheel displacement (cm). The last two equations are the wind "generator" and the coordinates z_6, z_7 are the horizontal and vertical components of the wind speed (m/sec). The parameter u is at the disposal of the first player, and the parameter $v = (v_1, v_2)$ at the disposal of the second. Let M be the hexagon in the z_1, z_2 plane with vertices $(-3,0)$, $(-3,1)$, $(0,1)$, $(3,0)$, $(3, -1)$, and $(0, -1)$. We put $\gamma(z_1, z_2) = \min\{c \geqslant 0: (z_1, z_2)' \in cM\}$. The first player minimizes the pay-off $\gamma(z_1(\vartheta), z_2(\vartheta))$ at the instant of termination $\vartheta = 15$ sec, and the second maximizes it. We can best treat ϑ as the time of hitting the end of the runway.

Game problems concerned with aircraft control in the vertical plane on landing, have been considered in /11, 12/. As distinct from these papers, the pay-off function here depends on two coordinates, and not one, of the phase vector.

In game (6.1), set P is an interval and Q is a rectangle. We take $P^2 = P$, $\gamma^2 = \gamma$. Let $\varkappa = 0.05$.

When checking sections $\mathrm{W}(t_i, c)$, we assumed that $\mathrm{L}(t_i, c) = L(t_i, c)$. Hence $\mathrm{W}(t_i, c) = W(t_i, c)$. Sections $\mathrm{W}(t_i, c)$ are symmetric about zero. In Fig.2 we show the upper parts of the sections, obtained by computer with $c = 1$ for instants $t_i = 12, 13, 14, 15$. Since $\gamma^2 = \gamma$, then $M^2(c) = M$. Hence $\mathrm{W}(15, 1) = M^2(1) = M$. In Fig.3 the broken line gives the polyhedron $\mathrm{H}(t_i, c)$ for $c = 1, t_i = 13$. For comparison, the continuous line gives section $\mathrm{W}(t_i, c)$. We also show the cones appearing in $\Lambda(t_i, c)$.

Let us quote the results of numerical modelling of the motions of system (6.1) for the initial position $t_* = 0$, $z_* = (5, 0, 0, 0, 0, 0, 0)$. The game value in this position is 0.8.

Let V° be the second player's strategy corresponding to the algorithm with correction. For specifying the strategy we used information taken from the polyhedra $\mathrm{W}(t_i, c)$ with $c = 0.81$, 0.9, and 1. We took the parameter μ equal to 0.01, and the step Δ of the discrete scheme equal to 0.5. The initial value c_* is 0.81, and the initial vector l_* was found with the aid of the set $\mathrm{W}(0, c_*)$. When realizing strategy V° each coordinate of the state vector $z(\tau_k)$ of system (6.1) was rounded before choosing the control action to the first plate after the decimal point. In this way we simulated possible errors in the scheme for forming the wind disturbances.

The first player's optimal strategy U° is specified numerically by means of a switching surface /8/, constructed by processing the polyhedra $\mathrm{W}(t_i, c)$ with $c = 0.81, 0.9, 1, 2, 4, 6$. We introduce three ways of realizing U°. Method A is a realization with step $\Delta_u = 0.05$; to form the control we use exact information on the position $z(t_* + k\Delta_u)$, $k = 0, 1, 2, \ldots$. With method B the step Δ_u is 0.5; at each instant $t_* + k\Delta_u$ the control is formed on the basis of the information about the first five coordinates of the vector $z(t_* + k\Delta_u)$; the sixth and seventh coordinates are assumed to be unmeasurable, and instead of them we put zeros in the scheme for choosing the control. Method C differs from method B in that a delay of 0.2 is introduced into the control development. Method A is the most accurate, and C is the crudest.

In Fig.4 we show curves of the variation of $z_1(t)$ when the second player uses strategy V° and the first, strategy U°. The letter next to the curve indicates the method of the first player's control. The pay-off's γ at the instant of termination for methods A, B, C are 0.68, 2.84, and 5.55. The realization, corresponding to method A, of the wind speed horizontal and vertical components $z_6(t)$, $z_7(t)$ are shown in Fig.5

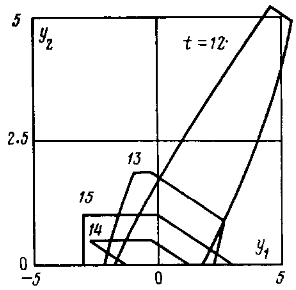

Fig.2

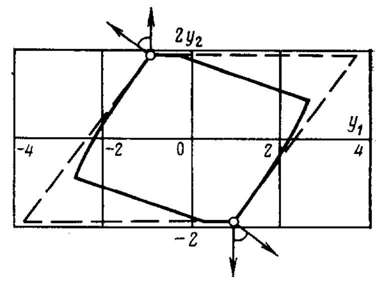

Fig.3

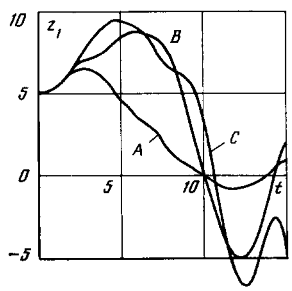

Fig.4

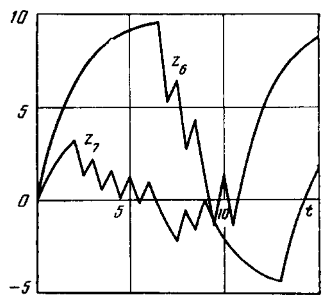

Fig.5

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